Extended tanh-Function Method for Finding Travelling Wave Solutions of Some Nonlinear Partial Differential Equations

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In this paper, we find travelling wave solutions of some nonlinear partial differential equations (PDEs) by using the extended tanh-function method. Some illustrative equations are investigated by this method and new travelling wave solutions are found. In addition, the properties of these nonlinear PDEs are shown with some figures.

Key words: tanh-Function Method; Nonlinear PDEs; Travelling Wave Solution; Symbolic Computation.

1. Introduction

The investigation of the travelling wave solutions plays an important role in nonlinear science. These solutions may well describe various phenomena in nature, such as vibrations, solitons and propagation with a finite speed. The wave phenomena are observed in fluid dynamics, in plasmas and in elastic media. Various methods for obtaining explicit travelling solitary wave solutions to nonlinear evolution equations have been proposed. In recent years, directly searching for exact solutions of nonlinear partial differential equations (PDEs) has become more and more attractive, partly due to the availability of computer symbolic systems like Maple or Mathematica which allow us to perform some complicated and tedious algebraic calculation on a computer, as well as help us to find new exact solutions of PDEs [1 – 8]. One of the most effectively straightforward methods to construct exact solutions of PDEs is the extended tanh-function method [9 – 13]. Recently, Elwakil et al. [14, 15] developed a modified extended tanh-function method for solving nonlinear PDEs.

Let us briefly describe the extended tanh-function method: Given a nonlinear equation

$$H(u,u_t,u_x,u_{xx},u_{xt},\ldots)=0. \tag{1}$$

We look for its travelling wave solutions, the first step is to introduce the wave transformation $u=U(\xi)$, $\xi=x+\lambda t$ and to change (1) to an ordinary differential equation

$$H\left(U,U',U'',\ldots\right)=0. \tag{2}$$

The next crucial step is to introduce a new variable $\varphi=\varphi(\xi)$ which is a solution of the Riccati equation

$$\varphi'=k+\varphi^2. \tag{3}$$

Then we propose the following series expansion as a solution of (1):

$$u(x,t)=U(\xi)=\sum_{i=0}^{m}a_i\varphi^i, \tag{4}$$

where the positive integer m can be determined by balancing the highest derivative term with the highest order nonlinear term in (2). Substituting (3) and (4) into (2) will result in a system of algebraic equations with respect to a_i,k, and λ (where $i=0,1,\ldots,m$) because all the coefficients of φ^i have to vanish. With the aid of Mathematica, one can determine a_i,k, and λ. The Riccati equation (3) has the general solutions

$$\varphi=\begin{cases}-\sqrt{-k}\tanh\left(\sqrt{-k}\xi\right)\\-\sqrt{-k}\coth\left(\sqrt{-k}\xi\right)\end{cases},\ \text{for } k<0, \tag{5}$$

$$\varphi=-\frac{1}{\xi},\ \text{for } k=0, \tag{6}$$

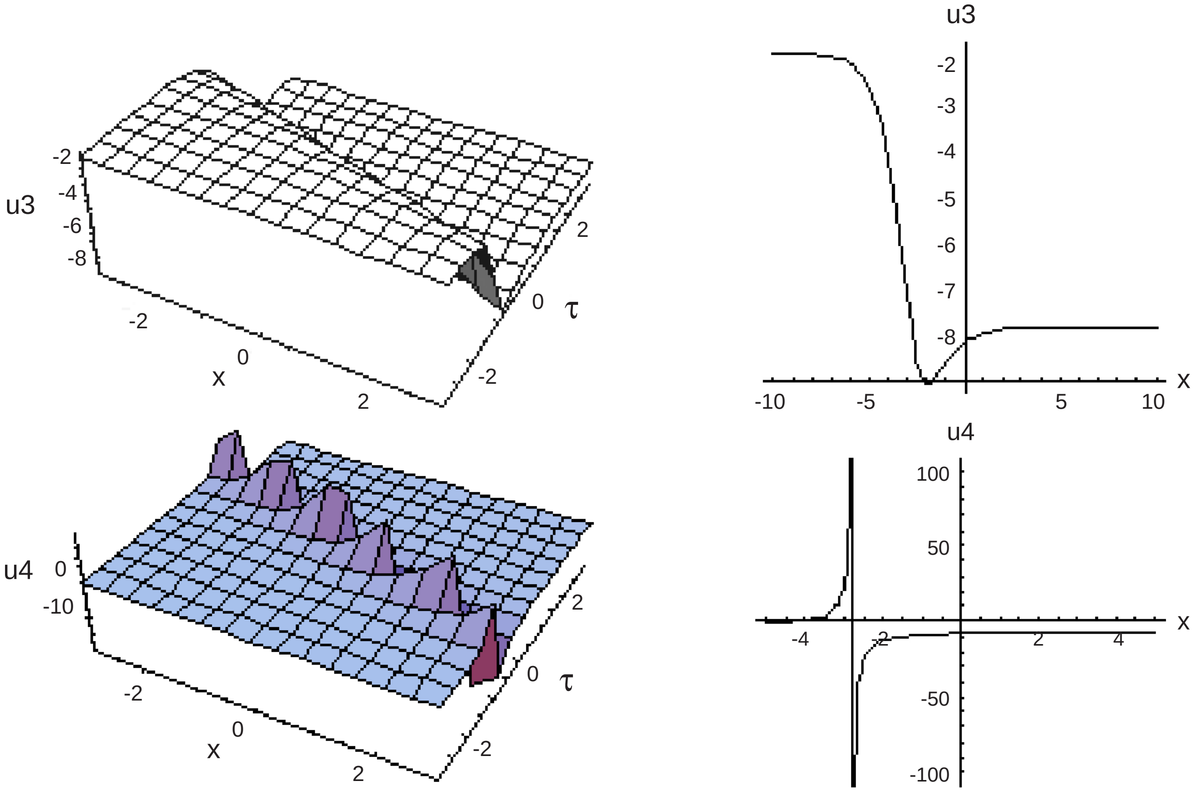

Fig. 1. The travelling wave solutions of u_3 and u_4, and its plots at $t = 1$, where $E = -0.5$, $k = -1$, $a_1 = 3$, and $\mu = -2$.

$$\varphi = \begin{cases} \sqrt{k}\tan\left(\sqrt{k}\xi\right) \\ -\sqrt{k}\cot\left(\sqrt{k}\xi\right) \end{cases}, \text{ for } k > 0. \tag{7}$$

In the next section, we study some nonlinear (1+1)-, (2+1)-, and (3+1)-dimensional nonlinear PDEs to illustrate this method.

2. Examples

2.1. Example of Nonlinear Interaction Model Equation

We consider the nonlinear interaction model of dispersion and dissipation [16, 17]:

$$u_t + E^2 u_x + (\partial_x - E)\,\partial_x\,[u_x - \mu u^n + Eu] = 0, \tag{8}$$

where E, μ are constants. Here we consider only the case $n = 2$. Let $u = U(\xi)$, $\xi = x - \lambda t$, where λ is the speed of propagating waves. Equation (8) is thus transformed to the nonlinear ordinary differential equation as

$$-\lambda U + E\mu U^2 + U'' - 2\mu U U' = F, \tag{9}$$

where F is an integration constant which is taken equal to zero.

Balancing U'' with U^2 yields $m = 2$. Therefore, we have

$$U = a_0 + a_1\varphi + a_2\varphi^2. \tag{10}$$

Substituting (10) into (9) and using Mathematica yields a set of algebraic equations for a_0, a_1, a_2, k, and λ:

$$\begin{aligned}
&-\lambda a_0 + E\mu a_0^2 + 2k^2 a_2 - 2k\mu a_0 a_1 = 0,\\
&-\lambda a_1 + 2E\mu a_0 a_1 + 2k a_1 - 4k\mu a_2 - 2k\mu a_1^2 = 0,\\
&-\lambda a_2 + 2E\mu a_0 a_2 + E\mu a_1^2 + 8k a_2\\
&\qquad - 2\mu a_0 a_1 - 6k\mu a_1 a_2 = 0,\\
&2E\mu a_1 a_2 + 2a_1 - 4\mu a_2 - 2\mu a_1^2 - 4k\mu a_2^2 = 0,\\
&E\mu a_2^2 + 6a_2 - 6\mu a_1 a_2 = 0,\\
&-4\mu a_2^2 = 0.
\end{aligned}$$

From the output of the symbolic computation soft-

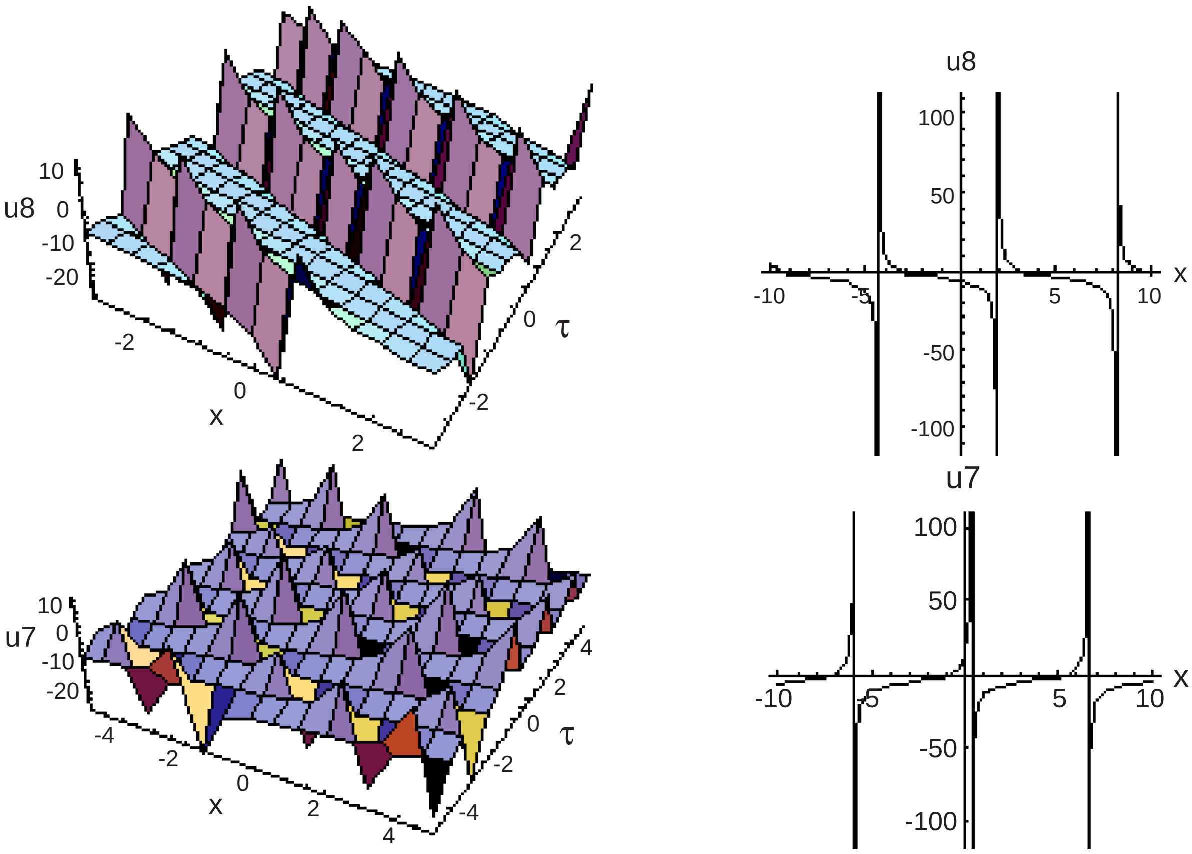

Fig. 2. The periodic wave solutions of u_7 and u_8, and its plots at $t = 1$, where $E = -0.5$, $k = -1$, $a_1 = 3$, and $\mu = -2$.

ware Mathematica we obtain

$$k = 0, a_0 = \frac{E}{2}a_1, a_2 = 0, \text{ and } \lambda = E^2\mu, \tag{11}$$

$$k \neq 0,\ a_0 = \frac{1}{E}\left(\frac{4ka_1}{1+\mu a_1}\right),\ a_2 = 0,\ \text{and} \quad \lambda = 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right), \tag{12}$$

where a_1 and k are arbitrary constants. Since k is an arbitrary parameter, according to (5)–(7), (11) and (12), we obtain three kinds of travelling wave solutions for Equation (9):

- Soliton solutions with $k < 0$:

$$u_1 = \frac{1}{E}\left(\frac{4ka_1}{1+\mu a_1}\right) - a_1\sqrt{-k} \cdot \tanh\left[\sqrt{-k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right], \tag{13}$$

$$u_2 = \frac{1}{E}\left(\frac{4ka_1}{1+\mu a_1}\right) - a_1\sqrt{-k} \cdot \coth\left[\sqrt{-k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right], \tag{14}$$

$$u_3 = \frac{1}{E}\left(\frac{4ka_1}{1+\mu a_1}\right) - a_1\sqrt{-k} \cdot \left\{\tanh\left[\sqrt{-k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right] + i\,\mathrm{sech}\left[\sqrt{-k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right]\right\}, \tag{15}$$

$$u_4 = \frac{1}{E}\left(\frac{4ka_1}{1+\mu a_1}\right) - a_1\sqrt{-k} \cdot \left\{\coth\left[\sqrt{-k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right] + \mathrm{csch}\left[\sqrt{-k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right]\right\}. \tag{16}$$

- Periodic solutions with $k > 0$:

$$u_5 = \frac{1}{E}\left(\frac{4ka_1}{1+\mu a_1}\right) + a_1\sqrt{k} \cdot \tan\left[\sqrt{k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right], \quad (17)$$

$$u_6 = \frac{1}{E}\left(\frac{4ka_1}{1+\mu a_1}\right) - a_1\sqrt{k} \cdot \cot\left[\sqrt{k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right], \quad (18)$$

$$u_7 = \frac{1}{E}\left(\frac{4ka_1}{1+\mu a_1}\right) - a_1\sqrt{k} \cdot \left\{\tan\left[\sqrt{k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right] - \sec\left[\sqrt{k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right]\right\}, \quad (19)$$

$$u_8 = \frac{1}{E}\left(\frac{4ka_1}{1+\mu a_1}\right) - a_1\sqrt{k} \cdot \left\{\cot\left[\sqrt{k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right] - \csc\left[\sqrt{k}\left(x - 2k\left(\frac{1-\mu a_1}{1+\mu a_1}\right)t\right)\right]\right\}. \quad (20)$$

- A rational solution with $k = 0$:

$$u_9 = \frac{E}{2}a_1 + \frac{1}{x - E^2\mu t}a_1. \quad (21)$$

Remark 1. It is easily seen that u_1, u_2, u_5, u_6 are similar to the solutions found already by Hu and Zhang [17]. But to our knowledge, the solutions u_3, u_4, u_7, u_8 of (8) (Figs. 1 and 2) have not been found before.

2.2. Example of Korteweg-de Vries-Burgers Equation

We now consider the compound Korteweg-de Vries-Burgers equation

$$u_t + puu_x + qu^2u_x + ru_{xx} - su_{xxx} = 0, \quad (22)$$

where p, q, r, s are constants. This equation can be thought of as a generalization of the Korteweg-de Vries (KdV), modified Korteweg-de Vries (mKdV) and Burgers equations, involving nonlinear dispersion and dissipation effects. Wang [18] has found some exact solutions of (22) by using the homogeneous balance method. Zheng et al. [19] have obtained new explicit and exact travelling wave solutions for (22) by using an improved sine-cosine method. Feng [20] proposed a new approach which was currently called the first integral method to study (22).

We assume formal solutions of the form

$$u = U(\xi),\ \xi = \lambda(x + bt + c_0), \quad (23)$$

where λ, b are constants to be determined later and c_0 is an arbitrary constant. Substituting (23) into (22), we obtain the ordinary differential equation

$$bU' - pUU' - qU^2U' - \lambda rU'' + s\lambda^2U''' = 0. \quad (24)$$

Integrating (24) once yields

$$bU - \frac{1}{2}pU^2 - \frac{1}{3}qU^3 - \lambda rU' + s\lambda^2U'' = 0, \quad (25)$$

where we have assumed that the integration constant is equal to zero.

Balancing U'' with U^3 gives $m = 1$, hence

$$U = a_0 + a_1\varphi. \quad (26)$$

Substituting (26) into (25) and making use of (3), we get a system of algebraic equations for a_0, a_1, k, λ, and b:

$$\begin{aligned} &ba_0 - \frac{1}{2}pa_0^2 - \frac{1}{3}qa_0^3 - \lambda ra_1k = 0,\\ &ba_1 - pa_0a_1 - qa_1a_0^2 + 2s\lambda^2a_1k = 0,\\ &-\frac{1}{2}pa_1^2 - qa_0a_1^2 - \lambda ra_1 = 0,\\ &-\frac{1}{3}qa_1^3 + 2s\lambda^2a_1 = 0. \end{aligned}$$

Solving the set of equations using Mathematica, we get three solutions

$$k = 0,\ a_1 = \mp\lambda\sqrt{\frac{6s}{q}},\ b = a_0(p + qa_0), \quad (27)$$

$$a_1 = \mp\lambda\sqrt{\frac{6s}{q}},\ b = a_0(p + qa_0) \mp 2s\lambda^3\sqrt{\frac{6s}{q}}k, \quad (28)$$

$$a_1 = \frac{2\lambda r}{p + 2qa_0},\ b = a_0(p + qa_0) + \frac{4s\lambda^3r}{p + 2qa_0}, \quad (29)$$

where a_0 and k are arbitrary constants. Since k is an arbitrary parameter, in analogy to (5)–(7) and (27)–(29), we obtain three kinds of travelling wave solutions for (22):

• A rational solution with $k = 0$:

$$u_1 = a_0 \mp \sqrt{\frac{6s}{q}} \frac{1}{\left(x - \left(pa_0 + qa_0^2\right) t + c_0\right)}. \tag{30}$$

• Soliton solutions with $k < 0$:

$$u_2 = a_0 \pm \lambda \sqrt{-\frac{6sk}{q}} \tanh\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 \mp 2s\lambda^3 \sqrt{\frac{6s}{q}} k\right) t + c_0\right)\right], \tag{31}$$

$$u_3 = a_0 \pm \lambda \sqrt{-\frac{6sk}{q}} \coth\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 \mp 2s\lambda^3 \sqrt{\frac{6s}{q}} k\right) t + c_0\right)\right], \tag{32}$$

$$u_4 = a_0 - \frac{2\lambda r}{p + 2qa_0} \sqrt{-k} \tanh\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 + \frac{4s\lambda^3 r}{p + 2qa_0}\right) t + c_0\right)\right], \tag{33}$$

$$u_5 = a_0 - \frac{2\lambda r}{p + 2qa_0} \sqrt{-k} \coth\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 + \frac{4s\lambda^3 r}{p + 2qa_0}\right) t + c_0\right)\right], \tag{34}$$

$$\begin{aligned} u_6 = a_0 \pm \lambda \sqrt{-\frac{6sk}{q}} \Bigg\{ & \tanh\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 \mp 2s\lambda^3 \sqrt{\frac{6s}{q}} k\right) t + c_0\right)\right] \\ & + \mathrm{i}\,\mathrm{sech}\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 \mp 2s\lambda^3 \sqrt{\frac{6s}{q}} k\right) t + c_0\right)\right] \Bigg\}, \end{aligned} \tag{35}$$

$$\begin{aligned} u_7 = a_0 \pm \lambda \sqrt{-\frac{6sk}{q}} \Bigg\{ & \coth\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 \mp 2s\lambda^3 \sqrt{\frac{6s}{q}} k\right) t + c_0\right)\right] \\ & + \mathrm{csch}\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 \mp 2s\lambda^3 \sqrt{\frac{6s}{q}} k\right) t + c_0\right)\right] \Bigg\}, \end{aligned} \tag{36}$$

$$\begin{aligned} u_8 = a_0 - \frac{2\lambda r}{p + 2qa_0} \sqrt{-k} \Bigg\{ & \tanh\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 + \frac{4s\lambda^3 r}{p + 2qa_0}\right) t + c_0\right)\right] \\ & + \mathrm{i}\,\mathrm{sech}\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 + \frac{4s\lambda^3 r}{p + 2qa_0}\right) t + c_0\right)\right] \Bigg\}, \end{aligned} \tag{37}$$

$$\begin{aligned} u_9 = a_0 - \frac{2\lambda r}{p + 2qa_0} \sqrt{-k} \Bigg\{ & \coth\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 + \frac{4s\lambda^3 r}{p + 2qa_0}\right) t + c_0\right)\right] \\ & + \mathrm{csch}\left[\sqrt{-k}\lambda \left(x - \left(pa_0 + qa_0^2 + \frac{4s\lambda^3 r}{p + 2qa_0}\right) t + c_0\right)\right] \Bigg\}. \end{aligned} \tag{38}$$

• Periodic solutions with $k > 0$:

$$u_{10} = a_0 \mp \lambda \sqrt{\frac{6sk}{q}} \tan\left[\sqrt{k}\lambda \left(x - \left(pa_0 + qa_0^2 \mp 2s\lambda^3 \sqrt{\frac{6s}{q}} k\right) t + c_0\right)\right], \tag{39}$$

$$u_{11} = a_0 \pm \lambda \sqrt{\frac{6sk}{q}} \cot\left[\sqrt{k}\lambda \left(x - \left(pa_0 + qa_0^2 \mp 2s\lambda^3 \sqrt{\frac{6s}{q}} k\right) t + c_0\right)\right], \tag{40}$$

$$u_{12} = a_0 - \frac{2\lambda r}{p+2qa_0}\sqrt{k}\tan\left[\sqrt{k}\lambda\left(x-\left(pa_0+qa_0^2+\frac{4s\lambda^3 r}{p+2qa_0}\right)t+c_0\right)\right], \tag{41}$$

$$u_{13} = a_0 + \frac{2\lambda r}{p+2qa_0}\sqrt{k}\cot\left[\sqrt{k}\lambda\left(x-\left(pa_0+qa_0^2+\frac{4s\lambda^3 r}{p+2qa_0}\right)t+c_0\right)\right], \tag{42}$$

$$\begin{aligned} u_{14} = a_0 \pm \lambda\sqrt{-\frac{6sk}{q}}\Bigg\{ &\tan\left[\sqrt{-k}\left(x-\left(pa_0+qa_0^2 \mp 2s\lambda^3\sqrt{\frac{6s}{q}}k\right)t+c_0\right)\right] \\ &- \sec\left[\sqrt{-k}\lambda\left(x-\left(pa_0+qa_0^2 \mp 2s\lambda^3\sqrt{\frac{6s}{q}}k\right)t+c_0\right)\right]\Bigg\}, \end{aligned} \tag{43}$$

$$\begin{aligned} u_{15} = a_0 \pm \lambda\sqrt{-\frac{6sk}{q}}\Bigg\{ &\left[\sqrt{-k}\lambda\left(x-\left(pa_0+qa_0^2 \mp 2s\lambda^3\sqrt{\frac{6s}{q}}k\right)t+c_0\right)\right] \\ &- \csc\left[\sqrt{-k}\lambda\left(x-\left(pa_0+qa_0^2 \mp 2s\lambda^3\sqrt{\frac{6s}{q}}k\right)t+c_0\right)\right]\Bigg\}, \end{aligned} \tag{44}$$

$$\begin{aligned} u_{16} = a_0 - \frac{2\lambda r}{p+2qa_0}\sqrt{-k}\Bigg\{ &\tan\left[\sqrt{-k}\lambda\left(x-\left(pa_0+qa_0^2+\frac{4s\lambda^3 r}{p+2qa_0}\right)t+c_0\right)\right] \\ &- \sec\left[\sqrt{-k}\lambda\left(x-\left(pa_0+qa_0^2+\frac{4s\lambda^3 r}{p+2qa_0}\right)t+c_0\right)\right]\Bigg\}, \end{aligned} \tag{45}$$

$$\begin{aligned} u_{17} = a_0 - \frac{2\lambda r}{p+2qa_0}\sqrt{-k}\Bigg\{ &\left[\sqrt{-k}\lambda\left(x-\left(pa_0+qa_0^2+\frac{4s\lambda^3 r}{p+2qa_0}\right)t+c_0\right)\right] \\ &- \csc\left[\sqrt{-k}\lambda\left(x-\left(pa_0+qa_0^2+\frac{4s\lambda^3 r}{p+2qa_0}\right)t+c_0\right)\right]\Bigg\}. \end{aligned} \tag{46}$$

Remark 2. Wang [18], Zheng et al. [19], and Feng [20] have obtained many exact solutions of this equation with the methods presented in [18–20]. It is easily seen that u_2, u_3, u_4, u_5, u_{10}, u_{11}, u_{12}, u_{13} are similar to those of [18–20]. But to our knowledge, the solutions u_6, u_7, u_8, u_9, u_{14}, u_{15}, u_{16}, u_{17} of (22) have not been found before.

2.3. Example of Boussinesq Equation

We consider a (2+1)-dimensional generalization of Boussinesq equation [21, 22]

$$u_{tt} - u_{xx} - u_{yy} - \left(u^2\right)_{xx} - u_{xxxx} = 0, \tag{47}$$

and seek travelling wave solutions of the form

$$u(x,y,t) = U(\xi),\ \xi = \alpha x + \beta y - \lambda t, \tag{48}$$

where α, β, and λ are constants. Substituting (48) into (47), we get

$$\alpha^4 U'' + \alpha^2 U^2 + \left(\alpha^2+\beta^2-\lambda^2\right)U = 0. \tag{49}$$

Balancing U'' with U^2 we get $m = 2$ so that now

$$U = a_0 + a_1\varphi + a_2\varphi^2. \tag{50}$$

The relations we get among the parameters are given as follows:

$$\begin{aligned} &2\alpha^4 k^2 a_2 + \alpha^2 a_0^2 + \left(\alpha^2+\beta^2-\lambda^2\right)a_0 = 0, \\ &2\alpha^2 a_0 a_1 + \left(\alpha^2+\beta^2-\lambda^2\right)a_1 = 0, \\ &8\alpha^4 k a_2 + \alpha^2 a_1^2 + 2\alpha^2 a_0 a_2 + \left(\alpha^2+\beta^2-\lambda^2\right)a_2 = 0, \\ &2\alpha^2 a_1 a_2 = 0, \\ &\alpha^2 a_2^2 + 6\alpha^4 a_2 = 0. \end{aligned}$$

This set of equations is solved by

$$\begin{aligned} &k = 0,\ a_0 = 0,\ a_1 = -\sqrt{6}\alpha^2 k,\ a_2 = -6\alpha^2, \\ &\lambda = \left(\alpha^2+\beta^2\right)^{1/2}, \end{aligned} \tag{51}$$

$$\begin{aligned} &a_0 = -\sqrt{6}\alpha^2 k,\ a_1 = 0,\ a_2 = -6\alpha^2, \\ &\lambda = \left(\alpha^2+\beta^2-2\sqrt{6}\alpha^4 k\right)^{1/2}, \end{aligned} \tag{52}$$

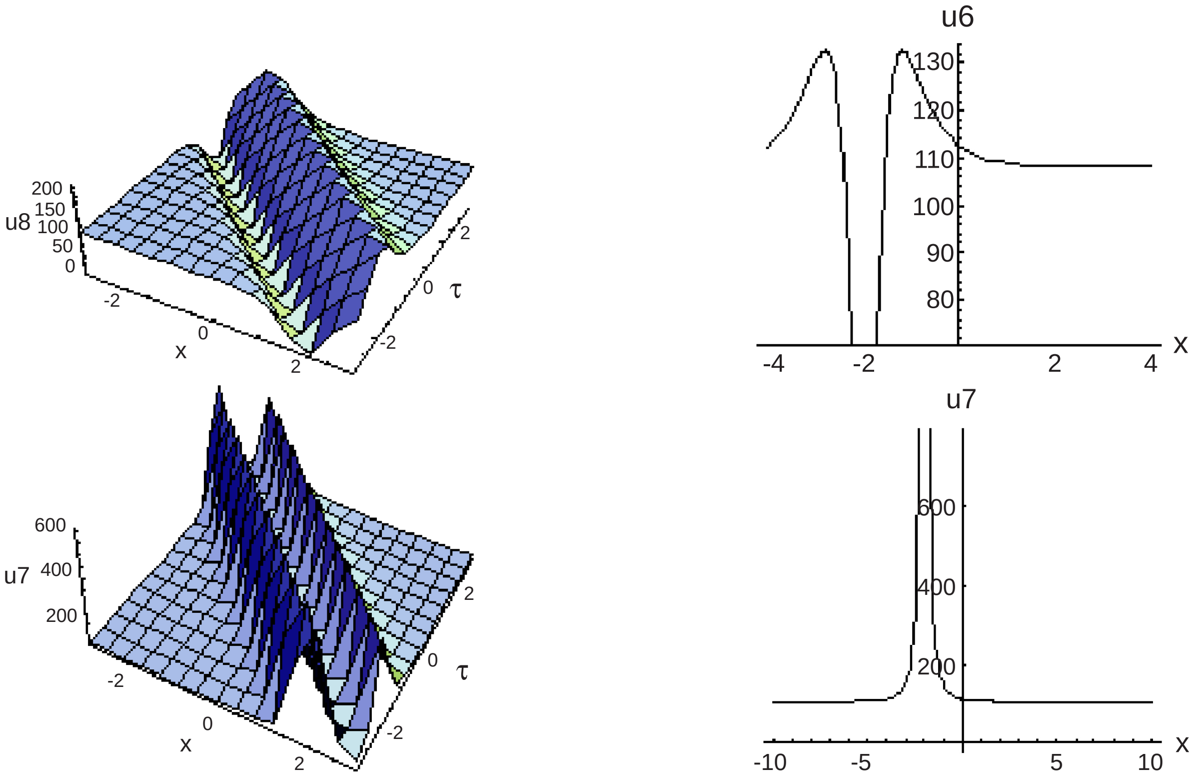

Fig. 3. The travelling wave solutions of u_6 and u_7, and its plots at $t=-1$, where $\alpha=\beta=1$, $\lambda=-1$, $k=-4$, and $a=3$.

$$a_0=\mp 2\sqrt{3}\mathrm{i}\alpha^2 k,\ a_1=0,\ a_2=-6\alpha^2,$$
$$\lambda=\left(\alpha^2+\beta^2\mp 4\sqrt{3}\mathrm{i}\alpha^4 k\right)^{1/2}. \tag{53}$$

Since k is an arbitrary parameter, according to (5)–(7) and (39)–(41), we obtain three kinds of travelling wave solutions for (47):

- A rational solution with $k=0$:

$$u_1=-\frac{\sqrt{6}\alpha^2 k}{\alpha x+\beta y-\left((\alpha^2+\beta^2)^{1/2}\right)t}-\frac{6\alpha^2}{\left(\alpha x+\beta y-\left((\alpha^2+\beta^2)^{1/2}\right)t\right)^2}. \tag{54}$$

- Soliton solutions with $k<0$:

$$u_2=-\sqrt{6}\alpha^2 k+6\alpha^2\sqrt{-k}\tanh^2\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right], \tag{55}$$

$$u_3=-\sqrt{6}\alpha^2 k+6\alpha^2\sqrt{-k}\coth^2\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right], \tag{56}$$

$$u_4=\mp 2\sqrt{3}\mathrm{i}\alpha^2 k+6\alpha^2\sqrt{-k}\tanh^2\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right], \tag{57}$$

$$u_5=\mp 2\sqrt{3}\mathrm{i}\alpha^2 k+6\alpha^2\sqrt{-k}\coth^2\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right], \tag{58}$$

$$u_6=-\sqrt{6}\alpha^2 k+6\alpha^2\sqrt{-k}\Big\{\tanh^2\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right] \mp \mathrm{i}\tanh\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right]\mathrm{sech}\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right]\Big\}, \tag{59}$$

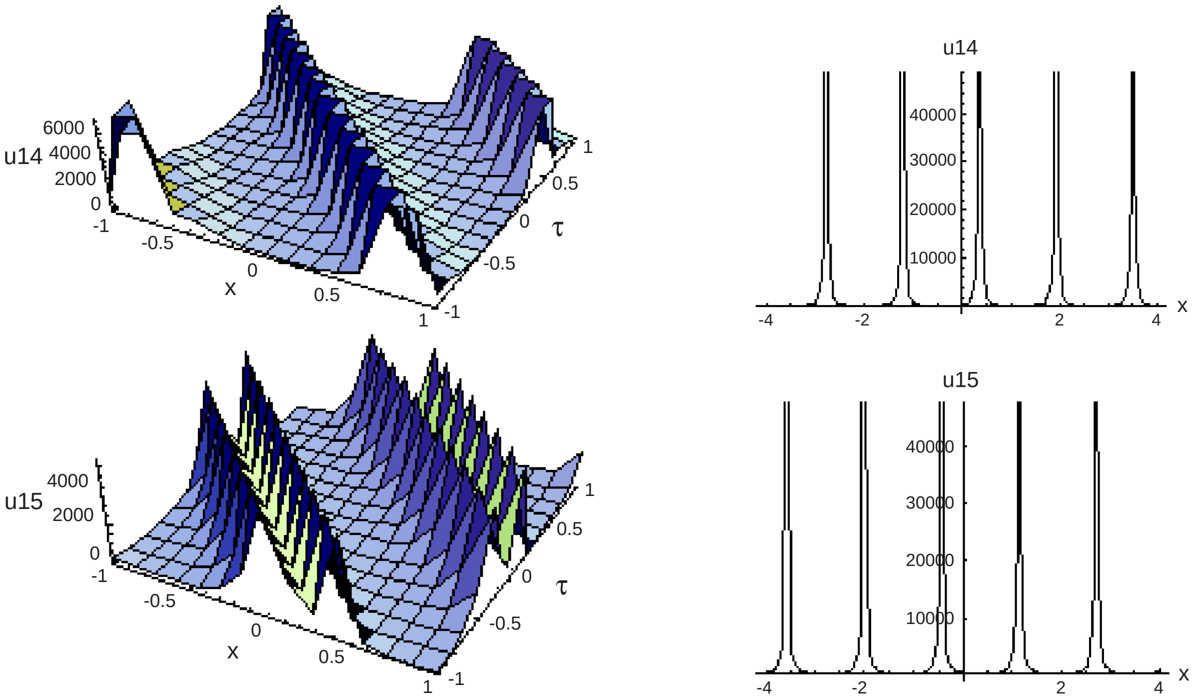

Fig. 4. The periodic wave solutions u_{14} and u_{15}, and its plots at $t=-1$, where $\alpha=\beta=1$, $\lambda=-1$, $k=-4$, and $a=3$.

$$u_7 = -\sqrt{6}\alpha^2 k + 6\alpha^2\sqrt{-k}\Big\{\coth^2\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right] \mp \coth\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right]\operatorname{csch}\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right]\Big\}, \tag{60}$$

$$u_8 = \mp 2\sqrt{3}\mathrm{i}\alpha^2 k + 6\alpha^2\sqrt{-k}\Big\{\tanh^2\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right] \mp i\tanh\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right]\operatorname{sech}\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right]\Big\}, \tag{61}$$

$$u_9 = \mp 2\sqrt{3}\mathrm{i}\alpha^2 k + 6\alpha^2\sqrt{-k}\Big\{\coth^2\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right] \mp \coth\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right]\operatorname{csch}\left[\sqrt{-k}(\alpha x+\beta y-\lambda t)\right]\Big\}, \tag{62}$$

where $\lambda = \left(\alpha^2+\beta^2-2\sqrt{6}\alpha^4 k\right)^{1/2}$.

- Periodic solutions with $k>0$:

$$u_{10} = -\sqrt{6}\alpha^2 k - 6\alpha^2\sqrt{k}\tan^2\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right], \tag{63}$$

$$u_{11} = -\sqrt{6}\alpha^2 k + 6\alpha^2\sqrt{k}\cot^2\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right], \tag{64}$$

$$u_{12} = \mp 2\sqrt{3}\mathrm{i}\alpha^2 k - 6\alpha^2\sqrt{k}\tan^2\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right], \tag{65}$$

$$u_{13} = \mp 2\sqrt{3}\mathrm{i}\alpha^2 k + 6\alpha^2\sqrt{k}\cot^2\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right], \tag{66}$$

$$u_{14} = -\sqrt{6}\alpha^2 k + 6\alpha^2\sqrt{k}\left\{\tan^2\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right] \mp \tan\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right]\sec\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right]\right\}, \tag{67}$$

$$u_{15}=-\sqrt{6}\alpha^2k+6\alpha^2\sqrt{k}\left\{\cot^2\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right]\mp\cot\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right]\csc\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right]\right\}, \quad (68)$$

$$u_{16}=\mp 2\sqrt{3}\mathrm{i}\alpha^2k+6\alpha^2\sqrt{k}\left\{\tan^2\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right]\mp\tan\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right]\sec\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right]\right\}, \quad (69)$$

$$u_{17}=\mp 2\sqrt{3}\mathrm{i}\alpha^2k+6\alpha^2\sqrt{k}\left\{\cot^2\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right]\mp\cot\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right]\csc\left[\sqrt{k}(\alpha x+\beta y-\lambda t)\right]\right\}, \quad (70)$$

where $\lambda=\left(\alpha^2+\beta^2-2\sqrt{6}\alpha^4k\right)^{1/2}$.

Remark 3. Senthilvelan [21] and Chen et al. [22] have obtained many exact solutions of this equation by the methods presented in [21, 22]. It is easily seen that u_2, u_3, u_4, u_5, u_{10}, u_{11}, u_{12}, u_{13} are similar to these solutions [21, 22]. But to our knowledge, the solutions u_6, u_7 (Fig. 3), u_8, u_9, u_{14}, u_{15} (Fig. 4), u_{16}, u_{17} of (47) have not been found before.

2.4. Example of Jumbo-Miwa Equation

We finally consider the (3+1)-dimensional equation of the form [21, 23]

$$u_{xxxy}+3u_{xy}u_x+3u_yu_{xx}+2u_{yt}-3u_{xz}=0, \quad (71)$$

which comes from the second member of a Kadomtsev-Petviashvili (KP) hierarchy called Jumbo-Miwa equation.

In this case we get an ODE of the form

$$\frac{\alpha^3\beta}{2}\left(U''\right)^2+\alpha^2\beta\left(U'\right)^3-\left(\beta\lambda+\frac{3\alpha}{2}\right)\left(U'\right)^2=0. \quad (72)$$

The balancing method yields

$$U=a_0+a_1\varphi, \quad (73)$$

where φ satisfies (3). As described in the previous sections we obtain the independent relations among the parameters as

$$a_1=-2\alpha,\ \lambda=-2\alpha^3k-\frac{3\alpha}{2\beta}, \quad (74)$$

$$a_1=\frac{4}{3}\alpha,\ \lambda=8\alpha^3k-\frac{3\alpha}{2\beta}, \quad (75)$$

where a_0 and k are arbitrary constants. Since k is an arbitrary parameter, according to (5)–(7), (74) and (75), we obtain two kinds of travelling wave solutions for (71):

- Soliton solutions with $k<0$:

$$u_1=a_0+2\alpha\sqrt{-k}\tanh\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right], \quad (76)$$

$$u_2=a_0+2\alpha\sqrt{-k}\coth\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right], \quad (77)$$

$$u_3=a_0-\frac{4}{3}\alpha\sqrt{-k}\tanh\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right], \quad (78)$$

$$u_4=a_0-\frac{4}{3}\alpha\sqrt{-k}\coth\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right], \quad (79)$$

$$u_5=a_0+2\alpha\sqrt{-k}\left\{\tanh\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right]+\mathrm{i}\,\mathrm{sech}\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right]\right\}, \quad (80)$$

$$u_6=a_0+2\alpha\sqrt{-k}\left\{\coth\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right]+\mathrm{csch}\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right]\right\}, \quad (81)$$

$$u_7=a_0-\frac{4}{3}\alpha\sqrt{-k}\left\{\tanh\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right]+\mathrm{i}\,\mathrm{sech}\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right]\right\}, \quad (82)$$

$$u_8=a_0-\frac{4}{3}\alpha\sqrt{-k}\left\{\coth\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right]+\mathrm{csch}\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right]\right\}. \quad (83)$$

where $\lambda=-\left(2\alpha^3k+3\alpha/2\beta\right)$.

- Periodic solutions with $k>0$:

$$u_9=a_0-2\alpha\sqrt{k}\tan\left[\sqrt{k}(\alpha x+\beta y+\lambda t)\right], \quad (84)$$

$$u_{10}=a_0+2\alpha\sqrt{k}\cot\left[\sqrt{k}(\alpha x+\beta y+\lambda t)\right], \quad (85)$$

$$u_{11}=a_0+\frac{4}{3}\alpha\sqrt{k}\tan\left[\sqrt{k}(\alpha x+\beta y+\lambda t)\right], \quad (86)$$

$$u_{12}=a_0-\frac{4}{3}\alpha\sqrt{k}\cot\left[\sqrt{k}(\alpha x+\beta y+\lambda t)\right], \quad (87)$$

$$u_{13} = a_0 + 2\alpha\sqrt{k}\left\{\tan\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right] - \sec\left[\sqrt{k}(\alpha x+\beta y+\lambda t)\right]\right\}, \quad (88)$$

$$u_{14} = a_0 + 2\alpha\sqrt{k}\left\{\cot\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right] + \csc\left[\sqrt{k}(\alpha x+\beta y+\lambda t)\right]\right\}, \quad (89)$$

$$u_{15} = a_0 - \frac{4}{3}\alpha\sqrt{k}\left\{\tan\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right] - \sec\left[\sqrt{k}(\alpha x+\beta y+\lambda t)\right]\right\}, \quad (90)$$

$$u_{16} = a_0 - \frac{4}{3}\alpha\sqrt{k}\left\{\cot\left[\sqrt{-k}(\alpha x+\beta y+\lambda t)\right] + \csc\left[\sqrt{k}(\alpha x+\beta y+\lambda t)\right]\right\}. \quad (91)$$

Remark 4. Senthilvelan [21] and Lu and Zhang [23] have already obtained many exact solutions of this equation by the methods presented in [21, 23]. It is easily seen that u_1, u_2, u_3, u_4, u_9, u_{10}, u_{11}, u_{12} are similar to those, see [21, 23]. But to our knowledge, the solutions u_5, u_6, u_7, u_8, u_{13}, u_{14}, u_{15}, u_{16} of (71) have not been found before. In addition, the properties of the some above solutions are shown in Figures 1 – 4.

3. Summary and Discussion

We have proposed an extended tanh-function method and used it to solve some (1+1)-, (2+1)-, and (3+1)-dimensional nonlinear PDEs. This method is readily applicable to a large variety of nonlinear PDEs. In addition, this method is computerizable, which allows us to perform complicated and tedious algebraic calculations on the computer. We have successfully recovered previously known solutions and also found new exact solutions, that previously have not been obtained by both the tanh-function and the homogeneous balance method. The travelling wave solutions derived in this article include soliton, periodic, and rational solutions.

The present method is direct and efficient to obtain more new exact and periodic wave solutions of (8), (22), (47), and (71). Our method is very easily applicable to nonlinear differential systems. The properties of the solutions are shown in Figures 1 – 4. The physical relevance of the soliton solutions and of the periodic solutions seems clear to us. We also can see that some solutions obtained in this article develop a singularity at a finite point in space, i. e. for any fixed $t = t_0$ there exists some x_0 at which these solutions blow up. There is much current interest in the formation of so-called “hot spots” or “blow up” of solutions [10, 24, 25]. Possibly these singular solutions are suitable to model physical phenomena.

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